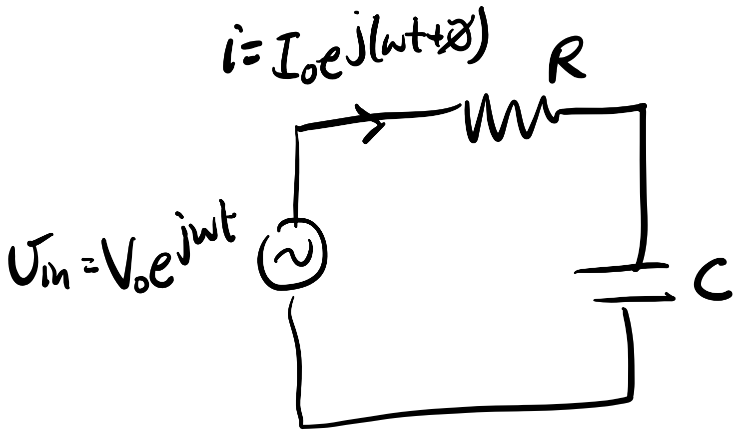
PHYS 231 – Oct. 23, 2023

<u>Last Time:</u> RC circuit



Found:

$$I_0 = \frac{\omega V_0 C}{\sqrt{1+(\omega RC)^2}}$$

$$\phi = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Useful result:

4 $$j\omega \underbrace{V_0 e^{j\omega t}}_{v_{in}} = \frac{I_0}{C}\underbrace{e^{j\omega t} e^{j\phi}}_{i} + j\omega R \underbrace{I_0 e^{j\omega t} e^{j\phi}}_{i}$$

$$j\omega v_{in} = \frac{i}{C} + j\omega R i$$

Divide by $j\omega$

$$v_{in} = \frac{i}{j\omega C} + iR$$

fcn gen. volt. | must be volt. across Capacitor | volt. across R

$$v_{in} = i\left(\frac{1}{j\omega C} + R\right)$$

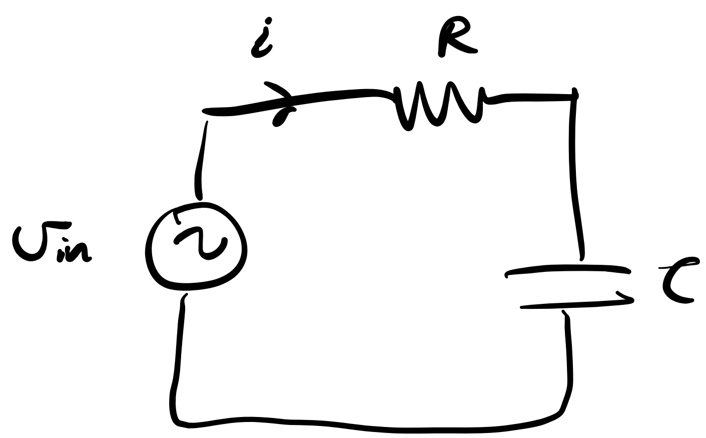
Now have volt. across resistor is usual $iR = v_R$

impedance of the resistor is $Z_R = R$

& volt. across cap. is $\underbrace{i Z_c}_{\text{impedance of capacitor.}} = v_c$

$$\boxed{Z_c = \frac{1}{j\omega C}}$$

For our RC circuit:



KVR :

$$v_{in} - v_R - v_c = 0$$

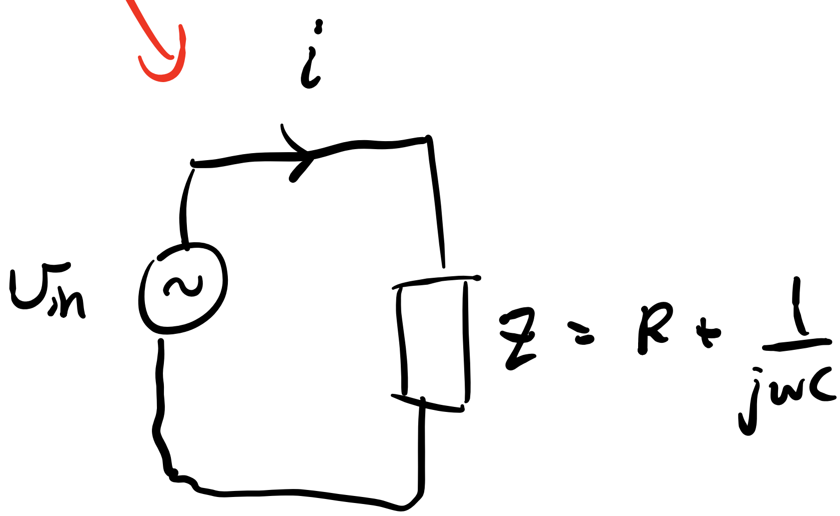
$$v_{in} - iR - iZ_c = 0$$

$$v_{in} = i(R + Z_c)$$

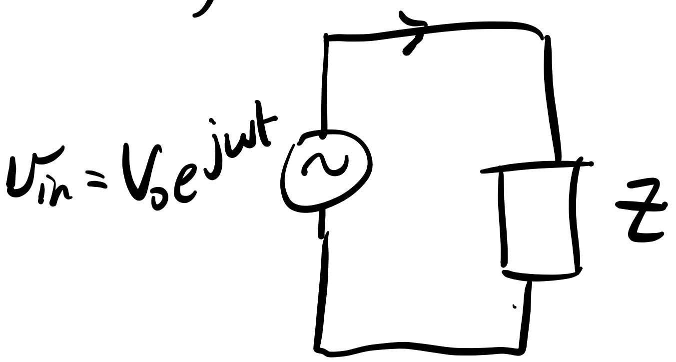
Can combine impedances in series & parallel in the same way we do for resistors.

Eg. series comb. of R & C gives an equiv. impedance of $Z = Z_R + Z_C$

$$= R + \frac{1}{j\omega C}$$



<u>Any</u> circuit w/ a single volt. source v_{in} & any comb. of R, C, L can be represented by $i = I_0 e^{j(\omega t + \phi)}$



<u>Goal:</u> Analyze this general circuit s.t. we can determine I_0 & ϕ.

$v_{in} = iZ$

$\therefore$

$i = \dfrac{v_{in}}{Z}$

replace $v_{in} = V_0 e^{j\omega t}$

$i = I_0 e^{j\omega t} e^{j\phi}$

Z is the net impedance of circuit.

Z in complex & any complex number can be written in the form $Z = |Z| e^{j\phi_Z}$

ϕ : phase of current.

ϕ_Z : phase of impedance Z. $\tan\phi_Z = \dfrac{\mathrm{Im}[Z]}{\mathrm{Re}[Z]} = \dfrac{y}{x}$

$$I_0 \cancel{e^{j\omega t}} e^{j\phi} = \frac{V_0 \cancel{e^{j\omega t}}}{|Z| e^{j\phi_Z}}$$

must be equal.

$$\underbrace{I_0}_{} e^{j\phi} = \underbrace{\frac{V_0}{|Z|}}_{} e^{-j\phi_Z}$$

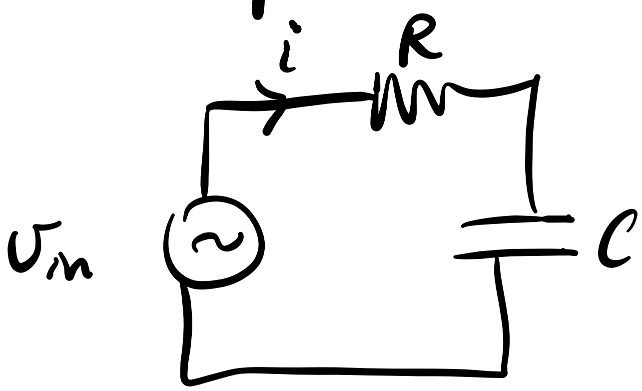
must be equal.

$$\boxed{\therefore\ I_0 = \frac{V_0}{|Z|}}$$

$$\boxed{\begin{aligned}\phi &= -\phi_Z \\ &= -\tan^{-1}\left(\frac{\mathrm{Im}[Z]}{\mathrm{Re}[Z]}\right)\end{aligned}}$$

Important Results for finding current.

Example: RC circuit.



know $I_0 = \dfrac{V_0}{|Z|}$

$$\phi = -\tan^{-1}\left(\frac{\mathrm{Im}[Z]}{\mathrm{Re}[Z]}\right)$$

$$Z = R + \underbrace{\frac{1}{j\omega C}}_{Z_C}\frac{j}{j} = R - \frac{j}{\omega C}$$

$x = R$

$y = -\dfrac{1}{\omega C}$

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \frac{1}{\omega C}\sqrt{1 + (\omega RC)^2}$$

$$\boxed{\therefore I_0 = \frac{V_0}{|Z|} = \frac{\omega V_0 C}{\sqrt{1+(\omega RC)^2}}}$$ same as before.

$$\tan \phi_z = \frac{y}{x} = \frac{-1/\omega C}{R} = -\frac{1}{\omega RC}$$

$$\therefore \phi_z = \tan^{-1}\left(-\frac{1}{\omega RC}\right) = -\tan^{-1}\left(\frac{1}{\omega RC}\right)$$

$$\boxed{\therefore \phi = -\phi_z = \tan^{-1}\left(\frac{1}{\omega RC}\right)}$$ same as before.

Impedance of an Inductor

Recall volt. across an inductor is given by

$$v_L = L\frac{di}{dt}$$

Express $i = I_0 e^{j(\omega t+\phi)}$

$$\therefore \frac{di}{dt} = j\omega \underbrace{I_0 e^{j(\omega t+\phi)}}_{i} = j\omega i$$

$$\therefore v_L = L(j\omega i) = i(j\omega L)$$

$$\therefore v_L = i \, Z_L$$

volt. across inductor

current in inductor

inductor impedance

$$\underline{\underline{Z_L = j\omega L}}$$

Note $[Z_L] = [Z_C] = [Z_R] = \Omega$

$$[\omega L] = \Omega$$

$$[L] = H = \frac{\Omega}{[\omega]} = \Omega s$$

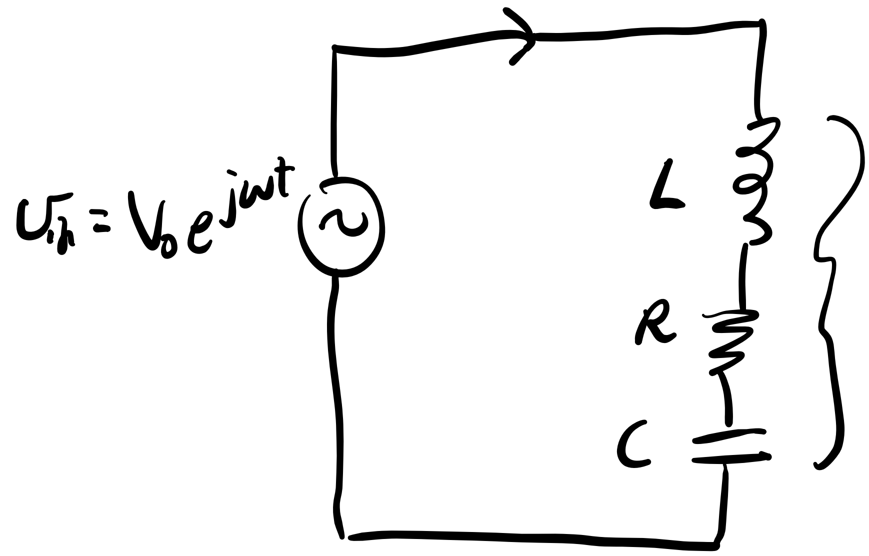
$$\left[\frac{1}{\omega C}\right] = \Omega$$

$$[C] = F = \frac{1}{\Omega[\omega]} = \frac{s}{\Omega}$$

$$[LC] = \cancel{\Omega} s \, \frac{s}{\cancel{\Omega}} = s^2$$

Current in an LRC circuit.

$$i = I_0 e^{j(\omega t + \phi)}$$



$$Z = Z_R + Z_L + Z_C$$

$$= R + j\omega L + \frac{1}{j\omega C}$$

$$\therefore Z = \underbrace{R}_{x} + j\underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{y}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = R\sqrt{1 + \left(\omega \frac{L}{R} - \frac{1}{\omega RC}\right)^2}$$

$$\tan \phi_Z = \frac{y}{x} = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R} = \omega \frac{L}{R} - \frac{1}{\omega RC}$$

$$\therefore\ \boxed{I_0 = \frac{V_0}{|Z|} = \frac{V_0/R}{\sqrt{1+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2}}}$$

$$\boxed{\phi = -\phi_Z = \tan^{-1}\left(\frac{1}{\omega RC}-\omega\frac{L}{R}\right)}$$